Supersymmetric transformations of real potentials on the line

1995 J. Phys. A: Math. Gen. 28 5079

(http://iopscience.iop.org/0305-4470/28/17/033)

J. Phys. A: Math. Gen. 28 (1995) 5079–5095. Printed in the UK

Supersymmetric transformations of real potentials on the line

J-M Sparenberg and D Baye

Physique Nucléaire Théorique et Physique Mathématique, CP 229, Université Libre de Bruxelles, Campus Plaine, B-1050 Brussels, Belgium

Received 9 May 1995

Abstract. A systematic study of supersymmetric (or Darboux) factorizations on the line is performed. All possible pairs of supersymmetric transformations with the same factorization energy are reviewed; for such pairs, there is no condition on this energy. Iterations of single transformations and of pairs both allow arbitrary modifications of the bound spectrum. Different iterative methods lead to compact analytical equations depending on the initial potential and its solutions. Iterations of single transformations are able to transform an even potential into an even potential. Particular cases of a method based on iteration of pairs conserve either the reflection coefficient at all energies or the norming constants of bound states. These results are compared with previous methods established in other contexts. The most general supersymmetric transformation of a given potential is finally described, both with and without modifications of the bound spectrum.

1. Introduction

In one-dimensional problems, supersymmetric quantum mechanics [1, 2] offers a simple way of constructing new potentials and their solutions in terms of a known potential and its solutions. Usually, some properties of the initial potential are modified by the transformation while other properties remain unchanged. For instance, when the new potentials are not singular, their bound spectra differ at most by the ground state from the initial spectrum.

For the line problem, many methods attempt to modify some properties of a potential, such as its spectrum, the norming constants of the bound states, or the asymptotic behaviour of the scattering states. Some of them [3–6] are directly based on supersymmetry. A number of methods [7–9] are based on the Darboux factorization, which is equivalent to a supersymmetric transformation. Other methods [6, 9–14] are based on general techniques of the inverse problem (see references in [15]). Although partial comparisons between these methods appear in these works, no theory brings all the aspects of the different methods together. The aim of the present paper is to establish a general framework, based on supersymmetry, which contains previous works as particular cases.

This theory is inspired by recent progress in the radial or half-line problem. About ten years ago, Sukumar [16] derived the possible supersymmetric transformations for this case, together with the corresponding modifications of the phase shifts. Applications often require potentials which share the same phase shifts at all energies, i.e. phase-equivalent potentials, but which have different bound spectra [17]. Such potentials, with an

0305-4470/95/175079+17$19.50

addition or a suppression of the ground state, can be obtained by pairs of supersymmetric transformations [17, 18]. Iterations of such pairs allow arbitrary modifications of the bound spectrum, without change of the phase shifts [19–21]. More importantly, non-singular phase-equivalent potentials can be obtained by a transformation pair which modifies the bound spectrum at another energy than the ground-state energy. In this case, the intermediate potential obtained after the first factorization is singular, but the singularity disappears in the final step [20, 22]. This property, which considerably reduces the difficulty of the problem, recently allowed us to derive the most general form of real potentials, phase equivalent to a given potential [23].

This success leads us to revisit supersymmetry for the line problem. In this context, phase equivalence is replaced by the conservation of reflection and transmission probabilities. This requirement is less restrictive than phase equivalence: it is fulfilled by *all* supersymmetric transformations, while phase equivalence in the radial case is only fulfilled by transformation pairs. A systematic study of supersymmetric transformations and of their combinations for the line problem is therefore timely.

Generally speaking, a supersymmetric transformation only requires energies where the initial Schrödinger equation is non-oscillatory, i.e. energies where solutions have a finite number of nodes (see definition in section XI.6 of [24]). Such a solution is called a factorization function, and its energy is called a factorization energy. Consequently, supersymmetric transformations can be applied to a very large class of potentials, such as confining potentials, oscillating potentials, or discontinuous potentials, since all these potentials can lead to non-oscillatory equations. However, in this article, we limit ourselves to rather regular potentials vanishing at infinity, namely the L_1^1 potentials (see the definition in equation (1)). The asymptotic behaviours of L_1^1-potential solutions, which are the key point of supersymmetric transformations, are exactly exponential functions. This simplifies the proofs of all the equations that follow. Let us note, however, that a more general theory, based on principal and non-principal solutions [24], instead of exponentially decreasing and increasing solutions, is possible. Moreover, solutions of oscillatory equations can also be used in supersymmetric transformations for studying bound states embedded in the continuum [22], but such bound states do not exist for L_1^1 potentials.

The factorization energy of a supersymmetric transformation has to be below the energy of the initial ground state in order to maintain the regularity of the potential. However, as in the radial case, this condition disappears when two successive transformations are applied at the same energy. A systematic study of transformation pairs is thus also useful for the line problem. Finally, with all single transformations and with all transformation pairs, iterations at different factorization energies can be performed in order to modify the bound spectrum arbitrarily. In these iterative methods, the type of single transformations or of transformation pairs leads to different properties, concerning for instance the potential symmetry, the norming constants of the bound states, or the reflection coefficients of the scattering states.

In section 2, we summarize some general characteristics of L_1^1 potentials and of their solutions. The principle of supersymmetric transformations is described in section 3. Pairs of supersymmetric transformations with the same factorization energy are studied in section 4. In sections 5 and 6, we iterate at different factorization energies single transformations and transformation pairs respectively. Finally, in section 7, we analyse different iterative methods, discuss their properties using a unicity theorem, and relate them to existing methods.

In what follows, subscripts refer to different Schrödinger equations, while superscripts between brackets refer in different ways to energies or wavenumbers.

2. General properties of the Schrödinger equation on the line

As mentioned in the introduction, we consider a real potential $V(x)$ in

$$L_1^1 \equiv \left\{ V(x) : \int_{-\infty}^{+\infty} |V(x)|(1+|x|)\,\mathrm{d}x < \infty \right\}. \tag{1}$$

This implies that $V(x)$ vanishes faster than x^{-2} at infinity, and that the physical or non-physical C^1 solutions $\varphi^{(E)}(x)$ of the Schrödinger equation

$$H\varphi^{(E)}(x) = \left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \right] \varphi^{(E)}(x) = E\varphi^{(E)}(x) \tag{2}$$

at energy E have real or complex exponential asymptotic behaviours [24]. Moreover, this implies that the bound spectrum is a finite set of N energies, $E_1 < \cdots < E_N < 0$ [7].

Using equation (2) for two arbitrary energies E and E', and for two arbitrary solutions $\varphi^{(E)}$ and $\chi^{(E')}$, a derivation and an integration of $W(\varphi^{(E)}, \chi^{(E')}) \equiv \varphi^{(E)}(\mathrm{d}\chi^{(E')}/\mathrm{d}x) - (\mathrm{d}\varphi^{(E)}/\mathrm{d}x)\chi^{(E')}$ lead to

$$W(\varphi^{(E)}, \chi^{(E')})(x) = W_0 + (E - E') \int_{x_0}^{x} \varphi^{(E)}\chi^{(E')}\,\mathrm{d}y. \tag{3}$$

When $E = E'$, the Wronskian is constant, and integrating (3) gives the general solution of (2), namely

$$\chi^{(E)}(x) = \varphi^{(E)}(x) \left[\chi^{(E)}(x_0)/\varphi^{(E)}(x_0) + W_0 \int_{x_0}^{x} (\varphi^{(E)}(y))^{-2}\,\mathrm{d}y \right]. \tag{4}$$

When $E \neq E'$, the Wronskian is not constant and can be expressed in integral form with (3).

For $E = -\kappa^2 < 0$ $(\kappa > 0)$, equation (2) has real solutions with real (increasing or decreasing) exponential asymptotic behaviours. When $E = E_j = -\kappa_j^2$ belongs to the bound spectrum, (2) has a unique normed physical solution $\psi^{(j)}(x)$ with

$$\lim_{x \to \pm\infty} \psi^{(j)}(x) \propto \exp(-\kappa_j |x|). \tag{5}$$

This represents a bound state of the system. Other solutions exist for the bound-spectrum energies, which do not vanish at infinity and which have no physical meaning, but we shall not need them in this article. A bound state is characterized by its norming constant

$$C^{(j)} = \lim_{x \to +\infty} \exp(\kappa_j x)|\psi^{(j)}(x)| \tag{6}$$

and the wavefunction has $j - 1$ nodes at finite distance.

For other negative energies, equation (2) has different types of non-normalizable solutions: *left* regular solutions $l^{(E)}(x)$, which satisfy

$$\lim_{x \to \pm\infty} l^{(E)}(x) \propto \exp(\kappa x) \tag{7}$$

right regular solutions $r^{(E)}(x)$, which satisfy

$$\lim_{x \to \pm\infty} r^{(E)}(x) \propto \exp(-\kappa x) \tag{8}$$

and *non-regular* solutions $n^{(E)}(x)$, which satisfy

$$\lim_{x \to \pm\infty} n^{(E)}(x) \propto \exp(\kappa |x|). \tag{9}$$

Actually, l- and r-solutions are defined up to a multiplicative constant, while n-solutions depend on two parameters. In this article, these two parameters are chosen as a 'shape parameter', which modifies the solution shape, and as a multiplicative constant. When E

belongs to $]E_{j-1}, E_j[$, l- and r-solutions have $j-1$ nodes, while n-solutions have j or $j-1$ nodes, following the shape-parameter value. Finally, when E is below the ground state, l- and r-solutions have no node, while n-solutions are nodeless or have a single node.

For $E = k^2 > 0$ $(k > 0)$, equation (2) has a unique oscillating complex solution $\psi^{(k)}(x)$ with the asymptotic behaviours

$$\lim_{x\to-\infty} \psi^{(k)}(x) = \exp(\mathrm{i}kx) + R^{(k)} \exp(-\mathrm{i}kx) \tag{10}$$

$$\lim_{x\to+\infty} \psi^{(k)}(x) = T^{(k)} \exp(\mathrm{i}kx). \tag{11}$$

The complex reflection and transmission coefficients $R^{(k)}$ and $T^{(k)}$ provide the reflection and transmission probabilities $|R^{(k)}|^2$ and $|T^{(k)}|^2$, respectively.

Deift and Trubowitz [7] have proved that a potential $V \in L_1^1$ is determined by its reflection coefficient $R^{(k)}$ (for all positive energies), its bound-state energies $E_1, \ldots, E_N$, and their norming constants $C^{(1)}, \ldots, C^{(N)}$. This implies that the transmission coefficient $T^{(k)}$ must also be defined by these elements. In fact, one has (p 154 of [7])

$$T^{(k)} = \lim_{\varepsilon\to0^+} \left[\exp\left(\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{+\infty} \frac{\ln(1-|R^{(\omega)}|^2)}{\omega - k - \mathrm{i}\varepsilon}\, \mathrm{d}\omega\right)\right] \prod_{j=1}^{N} (k + \mathrm{i}\kappa_j)/(k - \mathrm{i}\kappa_j) \tag{12}$$

where $R^{(-k)} = R^{(k)*}$. When V is a reflectionless potential, $T^{(k)}$ reduces to the final product.

3. Single supersymmetric transformations

Starting from Hamiltonian H_0 (i.e. from potential V_0), a new Hamiltonian H_1 (i.e. a new potential V_1) is constructed using a single supersymmetric transformation [2]. The Hamiltonian H_0 is factorized into

$$H_0 = A_0^+ A_0^- + \mathcal{E} \tag{13}$$

where $\mathcal{E} = -\epsilon^2$ $(\epsilon > 0)$ is the negative factorization energy. The operators $A_0^\pm$ are mutually adjoint and read

$$A_0^\pm = \pm\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x} \ln|\sigma_0^{(\mathcal{E})}| \tag{14}$$

where $\sigma_0^{(\mathcal{E})}$ is any real solution of the initial Schrödinger equation at energy $\mathcal{E}$. These operators only depend on one arbitrary parameter, since a multiplicative constant for $\sigma_0^{(\mathcal{E})}$ does not modify them. The second term of $A_0^\pm$ appearing in (14) is the most general form agreeing with (13).

The supersymmetric partner of H_0,

$$H_1 = A_0^- A_0^+ + \mathcal{E} \tag{15}$$

corresponds to a new real potential

$$V_1 = V_0 - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln|\sigma_0^{(\mathcal{E})}|. \tag{16}$$

In what follows, the supersymmetric transformation between H_0 and H_1 will be referred to as $T_\sigma^{(\mathcal{E})}$, where the subscript provides the type of factorization function, and the superscript refers to the factorization energy. Where no confusion is possible, the superscript will be omitted, both for the factorization function and for the supersymmetric transformation. The four types of solutions at negative energy correspond to four types of supersymmetric transformations: T_ψ, T_l, T_r, and T_n.

Applying A_0^- to the initial equation shows that the solutions of the new equation at energy E can be expressed as

$$\varphi_1^{(E)} = A_0^- \varphi_0^{(E)} = W(\varphi_0^{(E)}, \sigma_0^{(\mathcal{E})})/\sigma_0^{(\mathcal{E})}. \tag{17}$$

When $E \neq \mathcal{E}$, two linearly independent solutions of the initial equation lead to two linearly independent solutions of the new one. Moreover, the supersymmetric transformation conserves the solution types, as can be seen by examining the asymptotic behaviours of (17). From this it follows that the spectra of H_0 and H_1 are identical, except possibly for $\mathcal{E}$. However, when $\varphi_0^{(E)}$ and $\varphi_1^{(E)}$ are bound states ($E = E_j$), the normalization of $\psi_1^{(j)}$ imposes

$$\psi_1^{(j)} = |E_j - \mathcal{E}|^{-1/2} A_0^- \psi_0^{(j)}. \tag{18}$$

On the other hand, when $\varphi_0^{(E)}$ and $\varphi_1^{(E)}$ are scattering states ($E = k^2$), the required asymptotic behaviour (10) for $\psi_1^{(k)}$ at $-\infty$ imposes

$$\psi_1^{(k)} = \mathrm{i}(k + \mathrm{i}\epsilon)^{-1} A_0^- \psi_0^{(k)} \tag{19}$$

for σ_0 regular at $-\infty$, or

$$\psi_1^{(k)} = \mathrm{i}(k - \mathrm{i}\epsilon)^{-1} A_0^- \psi_0^{(k)} \tag{20}$$

for σ_0 singular at $-\infty$.

When $E = \mathcal{E}$, the Wronskian is constant and (17) only gives one solution of the new equation, proportional to σ_0^{-1}. Other solutions of the new equation at energy $\mathcal{E}$ are found using (4). For instance, for $\sigma_0 = \psi_0^{(\mathcal{E})}$, where $\mathcal{E}$ belongs to the spectrum of H_0, one has a solution

$$n_1^{(\mathcal{E})} = (\psi_0^{(\mathcal{E})})^{-1} \tag{21}$$

and a solution

$$\begin{aligned} l_1^{(\mathcal{E})} &= n_1^{(\mathcal{E})} \int_{-\infty}^{x} (n_1^{(\mathcal{E})})^{-2} \, \mathrm{d}y \\ &= (\psi_0^{(\mathcal{E})})^{-1} \int_{-\infty}^{x} (\psi_0^{(\mathcal{E})})^2 \, \mathrm{d}y \end{aligned} \tag{22}$$

which show that $\mathcal{E}$ does no longer belong to the bound spectrum of H_1: T_ψ removes $\mathcal{E}$ from the bound spectrum. Actually, the nature of $\mathcal{E}$ for the new equation is conditioned by the type of σ_0.

Things are more complicated in the case of T_n, which adds a bound state to the bound spectrum: the new bound state has to be normalized. As was seen in section 2, n-solutions depend on a shape parameter. They can be expressed in various ways, and some of them allow us to calculate the new bound-state norm easily. For instance, for

$$n_0^{(\mathcal{E})} = r_0^{(\mathcal{E})} \left[\alpha + \int_{-\infty}^{x} (r_0^{(\mathcal{E})})^{-2} \, \mathrm{d}y \right] \tag{23}$$

where the shape parameter α is strictly positive, the normalized wavefunction is given by

$$\psi_1^{(\mathcal{E})} = \alpha^{1/2} / n_0^{(\mathcal{E})}. \tag{24}$$

In the same way, T_l and T_r do not modify the bound spectrum; when $V_0 \equiv 0$, l- and r-solutions are simply exponential functions, so that V_1 is also 0.

Table 1 summarizes the properties of the four supersymmetric transformations. They lead to the following comments: (i) the theorem (12) is satisfied by $R_1^{(k)}$ and $T_1^{(k)}$; (ii) reflection and transmission probabilities are not affected by supersymmetric transformations;

Table 1. Single supersymmetric transformations.

Type	Spectrum modification	$R^{(k)}, T^{(k)}$	Norming constants
T_ψ	Suppression of $\mathcal{E}$	$R_1^{(k)} = -\dfrac{k-\mathrm{i}\epsilon}{k+\mathrm{i}\epsilon} R_0^{(k)}$	$C_1^{(j)} = \left\lvert\dfrac{\kappa_j - \epsilon}{\kappa_j + \epsilon}\right\rvert^{1/2} C_0^{(j)}$
		$T_1^{(k)} = \dfrac{k-\mathrm{i}\epsilon}{k+\mathrm{i}\epsilon} T_0^{(k)}$	$(E_j \neq \mathcal{E})$
T_l	No modification	$R_1^{(k)} = -\dfrac{k-\mathrm{i}\epsilon}{k+\mathrm{i}\epsilon} R_0^{(k)}$	$C_1^{(j)} = \left\lvert\dfrac{\kappa_j + \epsilon}{\kappa_j - \epsilon}\right\rvert^{1/2} C_0^{(j)}$
		$T_1^{(k)} = T_0^{(k)}$	$(E_j \neq \mathcal{E})$
T_r	No modification	$R_1^{(k)} = -\dfrac{k+\mathrm{i}\epsilon}{k-\mathrm{i}\epsilon} R_0^{(k)}$	$C_1^{(j)} = \left\lvert\dfrac{\kappa_j - \epsilon}{\kappa_j + \epsilon}\right\rvert^{1/2} C_0^{(j)}$
		$T_1^{(k)} = T_0^{(k)}$	$(E_j \neq \mathcal{E})$
T_n	Addition of $\mathcal{E}$	$R_1^{(k)} = -\dfrac{k+\mathrm{i}\epsilon}{k-\mathrm{i}\epsilon} R_0^{(k)}$	$C_1^{(j)} = \left\lvert\dfrac{\kappa_j + \epsilon}{\kappa_j - \epsilon}\right\rvert^{1/2} C_0^{(j)}$
		$T_1^{(k)} = \dfrac{k+\mathrm{i}\epsilon}{k-\mathrm{i}\epsilon} T_0^{(k)}$	$C_1^{(\mathcal{E})}$ arbitrary

(iii) in particular, a reflectionless potential remains reflectionless; (iv) for T_n, the norming constant $C_1^{(\mathcal{E})}$ can be arbitrarily chosen, for instance by fixing α in (23).

Let us finally discuss the problem of singularities in the potential due to supersymmetric transformations. The modification term in (16) vanishes faster than x^{-2} at infinity. Where σ_0 has nodes, it has non-integrable singularities. This can be seen using a series expansion. Hence, when V_0 is L_1^1, V_1 is L_1^1 if the factorization function σ_0 has no node, i.e. only when $\mathcal{E} \leqslant E_1$. In other cases, the supersymmetric transformation gives rise to a singular potential, and equation (17) shows that its solutions have singularities at the same locations. Actually, equations (17) to (24) remain valid between σ_0 nodes, since they are deduced from the local relation (17). Attention has only to be paid when an integral form is used, as for instance in (22). In these particular cases, the integral form is valid as long as the integral is well defined (which is true in (22), even if $\psi_0^{(\mathcal{E})}$ has nodes).

Conversely, a supersymmetric transformation can remove potential singularities when V_0 itself results from a supersymmetric transformation with factorization energy $\mathcal{E}$ larger than E_1. In what follows, we limit ourselves to L_1^1 potentials, but we sometimes use this kind of singular potentials in intermediate steps (an example appears in appendix A).

4. Pairs of supersymmetric transformations

Single supersymmetric transformations give rise to many different potentials. However, singularities occurring as soon as $\mathcal{E} > E_1$ apparently reduce the number of cases of physical interest. It has been recently shown, in the radial case [20–22], that some pairs of supersymmetric transformations, at the same factorization energy $\mathcal{E}$, can lead to regular potentials V_2, even when $\mathcal{E} > E_1$. In this case, intermediate potentials V_1 are singular at a finite number of points, and have no physical significance.

The same principle can be applied on the line, where the number of interesting cases is larger: only three pairs are useful in the radial case, whereas seven pairs occur here. The potential V_2, resulting from the pair $(T_\sigma^{(\mathcal{E})}, T_\tau^{(\mathcal{E})})$, i.e. from successive applications of $T_\sigma^{(\mathcal{E})}$

and $T_\tau^{(\mathcal{E})}$ on V_0, using (16) reads

$$V_2 = V_1 - 2\frac{d^2}{dx^2}\ln|\tau_1^{(\mathcal{E})}| = V_0 - 2\frac{d^2}{dx^2}\ln|\sigma_0^{(\mathcal{E})}\tau_1^{(\mathcal{E})}|. \tag{25}$$

Four types of single transformations lead to sixteen possible pairs. However, spectrum modifications make some pairs impossible: it is impossible to suppress or to add a bound state twice, so that (T_ψ, T_ψ) and (T_n, T_n) do not exist. In the same way, (T_l, T_ψ), (T_r, T_ψ), (T_n, T_l), and (T_n, T_r) do not exist, since $\mathcal{E}$ cannot be simultaneously physical and non-physical in the same spectrum. On the other hand, equation (25) shows that the pair of transformations does not modify the potential when τ_1 is proportional to σ_0^{-1}, as in (T_n, T_ψ), (T_l, T_r) and (T_r, T_l).

Of the sixteen pairs, six are impossible and three are trivial. The seven remaining pairs are all used in what follows. The potential V_2 and its solutions can be expressed in terms of V_0 and its solutions. Solutions φ_2 are expressed in terms of solutions φ_1 in the differential form (17). Solutions φ_1 are expressed in terms of solutions φ_0 in integral form, either deduced from (17)–(20) by using equation (3) for $E \neq \mathcal{E}$, or deduced from (4) as in (22) for $E = \mathcal{E}$. The validity of this integral form between σ_0 nodes has to be verified by direct calculation.

Let us introduce the integrals

$$\Phi(\varphi^{(E)}, \chi^{(E')}) = \int_{\pm\infty}^{x} \varphi^{(E)}\chi^{(E')}\,dy \tag{26}$$

where at least one energy is negative, and where the integration limit is chosen so that the integral converges. This is possible in all cases that are useful in what follows. When the two solutions are physical, both integration limits are allowed; in this article, we arbitrarily choose $-\infty$. When $E \neq E'$, these integrals are proportional to the corresponding Wronskians, as can be seen on (3) with $x_0 = \pm\infty$ and $W_0 = 0$. However, they are also defined when $E = E'$.

These Φ-functions lead to unified equations for the seven pairs (we treat (T_l, T_n) in detail in appendix A; other cases are similar):

$$V_2 = V_0 - 2\frac{d^2}{dx^2}\ln|\beta + \Phi(\sigma_0, \sigma_0)| \tag{27}$$

$$\varphi_2^{(E)} = N^{(E)}\left[\varphi_0^{(E)} - \frac{\sigma_0\Phi(\varphi_0^{(E)}, \sigma_0)}{\beta + \Phi(\sigma_0, \sigma_0)}\right] \tag{28}$$

where

$$\beta = \begin{cases} 0 & (T_l, T_l), (T_r, T_r), (T_\psi, T_l) \\ -1 & (T_\psi, T_r) \\ \alpha & (T_l, T_n) \\ -\alpha & (T_r, T_n) \\ \alpha/(1-\alpha) & (T_\psi, T_n) \end{cases} \tag{29}$$

in which α is a strictly positive parameter, to be defined only when $\mathcal{E}$ is a bound state of V_2, i.e. when $\tau_1 = n_1^{(\mathcal{E})}$. When $E \neq \mathcal{E}$, $\varphi_0^{(E)}$ and $\varphi_2^{(E)}$ in (28) are of the same type, and this type can only be ψ, l or r: equation (28) does not give n-solutions. On the other hand, when $E = \mathcal{E}$, equation (28) only gives the physical wavefunction $\psi_2^{(\mathcal{E})}$, and $\varphi_0^{(\mathcal{E})}$ has to be of the same type as σ_0.

The normalization constant $N^{(E)}$ equals 1, except in the following cases: for a bound state of V_2 at energy $\mathcal{E}$, where

$$N^{(\mathcal{E})} = \alpha^{-1/2}. \tag{30}$$

For a scattering state with σ_0 and τ_1 regular at $-\infty$, where

$$N^{(k)} = (k - \mathrm{i}\epsilon)/(k + \mathrm{i}\epsilon) \tag{31}$$

or with σ_0 and τ_1 singular at $-\infty$, where

$$N^{(k)} = (k + \mathrm{i}\epsilon)/(k - \mathrm{i}\epsilon). \tag{32}$$

Let us note that the Φ-functions used in (27) and (28) are well defined in all useful cases. Moreover, V_2 is L_1^1 and $\varphi_2^{(E)}$ is C^1. The seven pairs of supersymmetric transformations lead thus to physical potentials and to analytical expressions of their solutions, for any factorization energy $\mathcal{E} < 0$. This striking fact can also be directly established by replacing (27) and (28) in $H_2\varphi_2^{(E)} = E\varphi_2^{(E)}$.

Table 2. Pairs of supersymmetric transformations.

Type	Spectrum modification	$R^{(k)}, T^{(k)}$	Norming constants
(T_ψ, T_l)	Suppression of $\mathcal{E}$	$R_2^{(k)} = \left(\frac{k-\mathrm{i}\epsilon}{k+\mathrm{i}\epsilon}\right)^2 R_0^{(k)}$	$C_2^{(j)} = C_0^{(j)}$
		$T_2^{(k)} = \frac{k-\mathrm{i}\epsilon}{k+\mathrm{i}\epsilon} T_0^{(k)}$	$(E_j \neq \mathcal{E})$
(T_ψ, T_r)	Suppression of $\mathcal{E}$	$R_2^{(k)} = R_0^{(k)}$	$C_2^{(j)} = \left\lvert\frac{\kappa_j - \epsilon}{\kappa_j + \epsilon}\right\rvert C_0^{(j)}$
		$T_2^{(k)} = \frac{k-\mathrm{i}\epsilon}{k+\mathrm{i}\epsilon} T_0^{(k)}$	$(E_j \neq \mathcal{E})$
(T_l, T_l)	No modification	$R_2^{(k)} = \left(\frac{k-\mathrm{i}\epsilon}{k+\mathrm{i}\epsilon}\right)^2 R_0^{(k)}$	$C_2^{(j)} = \left\lvert\frac{\kappa_j + \epsilon}{\kappa_j - \epsilon}\right\rvert C_0^{(j)}$
		$T_2^{(k)} = T_0^{(k)}$	$(E_j \neq \mathcal{E})$
(T_r, T_r)	No modification	$R_2^{(k)} = \left(\frac{k+\mathrm{i}\epsilon}{k-\mathrm{i}\epsilon}\right)^2 R_0^{(k)}$	$C_2^{(j)} = \left\lvert\frac{\kappa_j - \epsilon}{\kappa_j + \epsilon}\right\rvert C_0^{(j)}$
		$T_2^{(k)} = T_0^{(k)}$	$(E_j \neq \mathcal{E})$
(T_ψ, T_n)	No modification	$R_2^{(k)} = R_0^{(k)}$	$C_2^{(j)} = C_0^{(j)}$
		$T_2^{(k)} = T_0^{(k)}$	$C_2^{(\mathcal{E})} = \alpha^{1/2} C_0^{(\mathcal{E})}$
(T_r, T_n)	Addition of $\mathcal{E}$	$R_2^{(k)} = \left(\frac{k+\mathrm{i}\epsilon}{k-\mathrm{i}\epsilon}\right)^2 R_0^{(k)}$	$C_2^{(j)} = C_0^{(j)}$
		$T_2^{(k)} = \frac{k+\mathrm{i}\epsilon}{k-\mathrm{i}\epsilon} T_0^{(k)}$	$C_2^{(\mathcal{E})} = \alpha^{-1/2}$
			$\times \lim_{x\to+\infty} \exp(\epsilon x)\lvert r_0^{(\mathcal{E})}\rvert$
(T_l, T_n)	Addition of $\mathcal{E}$	$R_2^{(k)} = R_0^{(k)}$	$C_2^{(j)} = \left\lvert\frac{\kappa_j + \epsilon}{\kappa_j - \epsilon}\right\rvert C_0^{(j)}$
		$T_2^{(k)} = \frac{k+\mathrm{i}\epsilon}{k-\mathrm{i}\epsilon} T_0^{(k)}$	$C_2^{(\mathcal{E})} = 2\epsilon\alpha^{1/2}$
			$\times \lim_{x\to+\infty} \exp(\epsilon x)/\lvert l_0^{(\mathcal{E})}\rvert$

Spectrum modifications, reflection and transmission coefficients, and norming constants, are gathered in table 2 for each pair of transformations. They can be obtained either from table 1, or by direct calculation with (28).

5. Iteration of single transformations

Let us consider a sequence of M single supersymmetric transformations, with distinct factorization energies $\mathcal{E}_1, \ldots, \mathcal{E}_M$, and with fatorization functions $\sigma_m \equiv \sigma_m^{(\mathcal{E}_{m+1})}$, such that the potential remains $L_1^!$ at each step. Except for $\mathcal{E}_1 \leqslant E_1$, this condition depends on the sequence of transformations and cannot be expressed simply. Successive potentials are linked together by single transformations (equation (16)) as

$$V_{m+1} = V_m - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln|\sigma_m| \tag{33}$$

with $m = 0, \ldots, M-1$, and σ_m nodeless.

On the other hand, a unified equation gives the link between solutions of two successive equations

$$\varphi_{m+1}^{(E)} = \mathcal{N}_{m+1}^{(E)} W(\varphi_m^{(E)}, \sigma_m)/\sigma_m. \tag{34}$$

When $E \neq \mathcal{E}_{m+1}$, $\varphi_m^{(E)}$ and $\varphi_{m+1}^{(E)}$ in (34) are of the same type. When $E = \mathcal{E}_{m+1}$, $\varphi_{m+1}^{(\mathcal{E}_{m+1})}$ is of type σ_m^{-1}, and the type of $\varphi_m^{(\mathcal{E}_{m+1})}$ must be different from the type of σ_m (see section 3). The normalization coefficient $\mathcal{N}_{m+1}^{(E)}$ equals 1, except in the following cases: when $E \neq \mathcal{E}_{m+1}$ corresponds to a bound state, $\mathcal{N}_{m+1}^{(j)} = |E_j - \mathcal{E}_{m+1}|^{-1/2}$ (see equation (18)); when $E = \mathcal{E}_{m+1}$ is a bound energy of V_{m+1}, the Wronskian W of (34) is constant and $\mathcal{N}_{m+1}^{(\mathcal{E}_{m+1})} = \|(n_m^{(\mathcal{E}_{m+1})})^{-1}\|^{-1/2} W^{-1}$; when E is a scattering state, $\mathcal{N}_{m+1}^{(k)} = \mathrm{i}/(k \pm \mathrm{i}\epsilon_{m+1})$ according to whether σ_m is regular or singular at $-\infty$ (see equations (19) and (20)).

Equations (33) and (34) are particularly useful, since they allow us to prove (see appendix B) that the final potential V_M and its solutions can be expressed in terms of M and $(M+1)$-dimensional Wronskians involving solutions of the initial equation only as

$$V_M = V_0 - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln|W(\sigma_0^{(\mathcal{E}_1)}, \ldots, \sigma_0^{(\mathcal{E}_M)})| \tag{35}$$

$$\varphi_M^{(E)} = \left(\prod_{m=0}^{M-1} \mathcal{N}_{m+1}^{(E)}\right) \frac{W(\varphi_0^{(E)}, \sigma_0^{(\mathcal{E}_1)}, \ldots, \sigma_0^{(\mathcal{E}_M)})}{W(\sigma_0^{(\mathcal{E}_1)}, \ldots, \sigma_0^{(\mathcal{E}_M)})} \tag{36}$$

where the Wronskian $W(f_1, \ldots, f_M)$ is the determinant of an M dimensional matrix whose elements read $W^{p,q} = \mathrm{d}^p f_q/\mathrm{d}x^p$ for $p = 0, \ldots, M-1$ and $q = 1, \ldots, M$, and $\sigma_0^{(\mathcal{E}_{m+1})}$ is of the same type as σ_m.

These equations, which generalize the results of [7], call for several remarks:

(i) If V_0 is even, other potentials V_{m+1} are also even when factorization functions are chosen symmetric; this is only possible with ψ- and n-functions, and requires particular values of the shape parameters in the n-functions. This ability of conserving symmetry in a controllable way will not occur in the other methods discussed below.

(ii) Supersymmetric transformations modify the number of nodes of solutions. Let us consider an example: $(T_\psi^{(E_1)}, T_l^{(E)})$ with $E_1 < E < E_2$, which removes the ground state at energy E_1 and then modifies the potential without modifying its spectrum. The solutions $\psi_0^{(1)}$ and $l_1^{(E)}$ have no node, so that V_1 and V_2 are regular. On the other hand, $l_0^{(E)}$ has one node at finite distance, as seen in section 2. The first transformation thus removes one node from this function. Consequently, it is not surprising that functions appearing in Wronskians of (35) and (36) have nodes. This is particularly dramatic with T_n transformations, where the shape parameter determines the number of nodes of the n-solution, and has to be chosen so that the potential remains regular. An important example of this phenomenon is given in subsection 7.4.

(iii) The result is unaffected by any permutation of supersymmetric transformations, since this simply corresponds to permutations of columns in determinants, and signs of determinants have no consequence. This shows that the potential can have singularities at intermediate steps, but recover regularity after the last transformation. A sufficient condition for the final potential to be regular is that there exists an order of application of the transformations for which the potential is regular at each step, as assumed in this section.

(iv) Following remark 2 in section 3 of [7], an alternative form of Wronskians of solutions exists. It only involves solutions and their first derivatives, and can be obtained using the initial equation (2).

6. Iteration of transformation pairs

Let us now consider a sequence of M supersymmetric transformation pairs, with distinct factorization energies $\mathcal{E}_1, \ldots, \mathcal{E}_M$, and with first factorization functions $\sigma_{2m} \equiv \sigma_{2m}^{(\mathcal{E}_{m+1})}$. There is no particular condition on these energies, unlike for iterations of single transformations, as seen in section 4. The potential remains $L_1^\dagger$ at each step. Potentials and their solutions are linked together by equations (27)–(32)

$$V_{2(m+1)} = V_{2m} - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln |\beta_{m+1} + \Phi(\sigma_{2m}, \sigma_{2m})| \tag{37}$$

$$\varphi_{2(m+1)}^{(E)} = N_{m+1}^{(E)} \left[\varphi_{2m}^{(E)} - \frac{\sigma_{2m} \Phi(\varphi_{2m}^{(E)}, \sigma_{2m})}{\beta_{m+1} + \Phi(\sigma_{2m}, \sigma_{2m})} \right] \tag{38}$$

for $m = 0, \ldots, M-1$. The subscripts of $N^{(E)}$ and β refer to the factorization energy $\mathcal{E}_{m+1}$.

As in the case of single transformations, these equations allow us to express (see appendix C) the final potential V_{2M} and its solutions in terms of the initial equation solutions only, as

$$V_{2M} = V_0 - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln |\det \mathbf{X}_0| \tag{39}$$

$$\varphi_{2M}^{(E)} = \left(\prod_{m=0}^{M-1} N_{m+1}^{(E)} \right) \det \mathbf{Y}_0(\varphi_0^{(E)}) / \det \mathbf{X}_0. \tag{40}$$

In these expressions, $\mathbf{X}_0$ is an M-dimensional matrix, whose elements read, for $p, q = 1, \ldots, M$,

$$X_0^{p,q} = \beta_p \delta_{pq} + \Phi(\sigma_0^{(\mathcal{E}_p)}, \sigma_0^{(\mathcal{E}_q)}) \tag{41}$$

and $\mathbf{Y}_0(\varphi_0^{(E)})$ is an $(M+1)$-dimensional matrix, whose elements read

$$Y_0^{p,q}(\varphi_0^{(E)}) = \begin{cases} X_0^{p,q} & (p, q > 0) \\ \sigma_0^{(\mathcal{E}_p)} & (p > 0,\ q = 0) \\ \Phi(\varphi_0^{(E)}, \sigma_0^{(\mathcal{E}_q)}) & (p = 0,\ q > 0) \\ \varphi_0^{(E)} & (p = q = 0). \end{cases} \tag{42}$$

In these equations, Φ-functions are defined in (26); $\sigma_0^{(\mathcal{E}_{m+1})}$ is of the same type as σ_{2m}; β and $N^{(E)}$ coefficients are given by equations (29) and (30)–(32), respectively. When E does not belong to the set of factorization energies, $\varphi_{2M}^{(E)}$ and $\varphi_0^{(E)}$ in (40) are of the same type and

cannot be of type n. When $E = \mathcal{E}_{m+1}$, equation (40) only gives the physical wavefunction $\psi_{2M}^{(\mathcal{E}_{m+1})}$, and $\varphi_0^{(\mathcal{E}_{m+1})}$ has to be chosen of the same type as σ_{2m}. In this particular case, $\det \mathbf{Y}_0(\varphi_0^{(\mathcal{E}_{m+1})})$ can be reduced to an M-dimensional determinant by subtracting row $m+1$ from the similar row 0.

As for single transformations (remark (iii) of section 5), determinants appearing in (39) and (40) simply express the commutativity of transformation pairs. On the other hand, transformation pairs behave differently from single transformations as far as symmetry is concerned: they do not conserve it since they contain T_l or T_r transformations (this is not the case for (T_ψ, T_n), but this pair has no effect on the potential when it conserves symmetry).

7. Applications and comparisons with existing works

7.1. The reflection-coefficient method

Without modifying the reflection coefficient $R^{(k)}$, (T_ψ, T_r) and (T_l, T_n) respectively suppress and add a bound state, while (T_ψ, T_n) modifies its norming constant. This can be seen in table 2. Iterating these three pairs constitutes the reflection-coefficient method (R-method), which leads to arbitrary modifications of the bound spectrum and of norming constants, without modification of the reflection coefficient at any energy. According to the unicity theorem of section 2, the R-method thus allows us to construct *all* L_1^1 potentials having the same reflection coefficient as a given L_1^1 potential. Let us note that the control of norming constants with this method is not simple: any spectrum modification changes all norming constants. Restoring them then requires a (T_ψ, T_n) transformation pair for each bound state. Actually, this pair has the advantage of modifying arbitrarily one norming constant without modifying the others.

Abraham and Moses [11] present an algorithmic method, based on the Gel'fand–Levitan equation, which allows the same modifications of a given potential as the R-method. The link between both methods is proved for ground-state deletion, addition and renormalization on the line in [25], and for ground-state deletion in the radial problem in [26].

Establishing the exact equivalence between the Abraham–Moses and R-methods closely follows the analogous discussion of [20, 23] (see below), and we shall not reproduce it here. Let us simply mention that the R-method gives explicit solutions of the Abraham–Moses equation, instead of algorithmic solutions. Pursey and Weber [27] find similar solutions of this equation in the radial case.

7.2. The norming-constant method

Without modifying the norming constants of other bound states, (T_ψ, T_l) and (T_r, T_n) respectively suppresses and adds a bound state, while (T_ψ, T_n) modifies its norming constant. This can be seen in table 2. Iterating these three pairs constitutes the norming-constant method (C-method), which leads to arbitrary modifications of the bound spectrum and of some norming constants, without modifying the other norming constants. It is particularly useful in the radial problem [23], where it presents the additional property of conserving phase shifts. Compared with the R-method, the C-method is useful when one needs to control the norming constants.

On the other hand, the C-method modifies the phase of the reflection coefficient but does not modify its modulus, as can be seen in section 3. The unicity theorem of section 2 does not warrant that the C-method provides all potentials sharing specific properties, except in the case of reflectionless potentials: in fact, the C-method allows the construction of

all L^1_1 *reflectionless* potentials, starting from one of them, with a total control of norming constants.

Let us note that the C- and R-methods are formally very close to one another (the roles of $+\infty$ and $-\infty$ are just inverted). In spite of this similarity, they have quite different properties, because of the asymmetry between $+\infty$ and $-\infty$, introduced by the basic definitions (6), (10), and (11) of the line problem. A formal link between the Abraham–Moses and C-methods is given for the ground state in [16] and for arbitrary modifications of the bound spectrum in [20, 23], in the case of the radial problem.

In [10], Pursey proposes a method based on the Marchenko equation (instead of the Gel'fand–Levitan equation used by Abraham and Moses), which allows the same modifications of a given potential as the C-method. The link between the Pursey and C-methods is made in [26] for a ground-state deletion in the radial case. A general proof of equivalence is very close to that of [20, 23]. We shall not reproduce it here. Let us only mention, as for the R-method, that the C-method leads to general analytical solutions of the Pursey equation, and that Pursey and Weber also find these solutions in [27] for the radial case.

7.3. Removing M bound states of a given potential

Let V_0 have at least M bound states. Three main methods can be used to remove bound states:

(i) $T_\psi^{(\mathcal{E}_1)}, \ldots, T_\psi^{(\mathcal{E}_M)}$, where $\mathcal{E}_m = E_m$. The final potential is regular if and only if the removed bound states are the lowest ones. The order of application (E_1 to E_M) is only important if one wants the potential to remain regular at each step. This method maintains a possible potential symmetry and is used by Deift and Trubowitz [7].

(ii) $(T_\psi^{(\mathcal{E}_1)}, T_r^{(\mathcal{E}_1)}), \ldots, (T_\psi^{(\mathcal{E}_M)}, T_r^{(\mathcal{E}_M)})$, where the $\mathcal{E}_m$ belong to the initial bound spectrum. This method conserves the reflection coefficient (R-method), and is equivalent to that of Abraham and Moses [11].

(iii) $(T_\psi^{(\mathcal{E}_1)}, T_l^{(\mathcal{E}_1)}), \ldots, (T_\psi^{(\mathcal{E}_M)}, T_l^{(\mathcal{E}_M)})$, where the $\mathcal{E}_m$ belong to the initial bound spectrum. This method conserves the norming constants of the remaining bound states (C-method), and is equivalent to that of Pursey [10].

For (ii) and (iii), the removed bound states cannot be the lowest ones and a permutation of energies does not make the potential singular at intermediate steps, unlike for (i). For each method a direct analytical equation exists (particular cases of sections 5 and 6). An analytical equation also exists when (ii) and (iii) are mixed (section 6). In contrast, when (i)-(iii) are mixed, no simple analytical equation is available.

7.4. Adding M bound states to a given potential

As for the preceeding case, three main methods exist:

(i) $T_n^{(\mathcal{E}_1)}, \ldots, T_n^{(\mathcal{E}_M)}$. Here, the validity condition on energies can be expressed as $E_1 > \mathcal{E}_1 > \ldots > \mathcal{E}_M$. Functions $n_0^{(\mathcal{E}_m)}$ appearing in Wronskians of (35) and (36) must be nodeless if m is odd, and have one node if m is even. This is a consequence of remark (ii) of section 5. This method is used in [3, 4, 7].

(ii) $(T_l^{(\mathcal{E}_1)}, T_n^{(\mathcal{E}_1)}), \ldots, (T_l^{(\mathcal{E}_M)}, T_n^{(\mathcal{E}_M)})$, where the $\mathcal{E}_m$ do not belong to the initial bound spectrum (R-method).

(iii) $(T_r^{(\mathcal{E}_1)}, T_n^{(\mathcal{E}_1)}), \ldots, (T_r^{(\mathcal{E}_M)}, T_n^{(\mathcal{E}_M)})$, where the $\mathcal{E}_m$ do not belong to the initial bound spectrum (C-method).

For methods (ii) and (iii), and for a mixing of all methods, the same comments as in

subsection 7.3 apply. In particular, for methods (ii) and (iii), the added bound states do not have to be below the initial ground state. In the same way, comments concerning the order of application and singularities are similar to those of subsection 7.3.

Method (i) has been used quite often in previous works in the case $V_0 \equiv 0$ [3,4,7]. The same result can be obtained by other methods [12, 13], and linked with vertex operators [6, 14]. This is found in a study of solitons with the Korteweg–de Vries equation, whose solutions are reflectionless potentials. In this context, the number of bound states and their norming constants have a direct physical meaning [14] (see also [26]).

7.5. Most general supersymmetric transformation of a given potential

An iteration of T_r or T_l transformations at a given energy gives rise to a countable infinity of different isospectral potentials (the fact that $R^{(k)}$ changes at each step ensures that these potentials are different). Non-physical negative energies form a continuum, so that the different potentials obtained from supersymmetry are not countable. In fact, the most general transformation of a given potential can be performed in three steps:

(i) deletion of all bound states of the initial potential, using method (i) of subsection 7.3;

(ii) arbitrary number of T_r or T_l transformations, for any negative energy (not only for removed energies or for energies of final bound states);

(iii) introduction of bound states of the final spectrum, using method (i) of subsection 7.4.

In some cases, this method is a particular case of iterative methods of sections 5 and 6, and a compact analytical result exists. For instance, the application of (T_r, T_r) at any non-physical energy does not require deletion and reintroduction of all bound states.

7.6. Isospectral potentials

The general process explained in subsection 7.5 can be specialized in order to construct isospectral potentials: the bound states introduced at step (iii) must have the same energies as the initial bound states removed at step (i). The class of isospectral potentials obtained from supersymmetry can be compared with existing results.

Keung *et al* [5] construct an M-parameter family of isospectral potentials with M bound states by first removing the M bound states with successive T_ψ, and then reintroducing them with successive T_n. This is equivalent to M successive (T_ψ, T_n) pairs, each of them introducing an arbitrary shape parameter, and performed in arbitrary order.

Khare and Sukhatme [9] generalize this method by considering three ways of deleting the ground state and three ways of reinserting it. They note that, among the nine possible resulting transformations, only five are distinct. In the present notation, the three suppression methods correspond to T_ψ, (T_ψ, T_r), (T_ψ, T_l), while the three addition methods correspond to T_n, (T_r, T_n), (T_l, T_n). Taking into account that (T_r, T_l) and (T_l, T_r) are trivial pairs (see section 4), one directly obtains five ways of renormalizing the ground state, namely (T_ψ, T_n), (T_ψ, T_l, T_n), (T_ψ, T_r, T_n), (T_ψ, T_l^2, T_n) and (T_ψ, T_r^2, T_n). These authors show that iterating these methods leads to a countable infinity of isospectral-potential families. In the present notation, this iteration corresponds to $(T_\psi, T_r, \ldots, T_r, T_n)$ or $(T_\psi, T_l, \ldots, T_l, T_n)$ at each energy of the bound spectrum, since (T_n, T_ψ) is trivial.

The number of isospectral potentials obtained from general supersymmetric transformations is larger than the result of [9], since T_r and T_l transformations can be used for *all* negative energies. Moreover, in some cases compact analytical equations exist.

8. Conclusion

On the line as in the radial case, supersymmetry is a powerful way of analytically constructing new potentials and their solutions, starting from a given potential and its solutions. The case of L_1^1 potentials is studied here, in order to simplify proofs and to use a unicity theorem. A generalization of supersymmetry to more general potentials is possible.

The properties of all single transformations and of all transformation pairs are established. The condition on the factorization energy disappears when a pair is used. The results, summarized in tables 1 and 2, show that supersymmetry is not able to modify the reflection and transmission probabilities, but can modify the bound spectrum, norming constants, and the phases of the reflection and transmission coefficients.

General iterative methods of single transformations and of transformation pairs are explored. Among them, a method based on single transformations (section 5) can maintain an even symmetry of the potential; the R-method (subsection 7.1) conserves the reflection coefficient at all energies and gives rise to *all* potentials having the same $R^{(k)}$ as a given potential; the C-method (subsection 7.2) conserves norming constants and allows the construction of *all reflectionless* potentials, with a total control of norming constants.

Examining supersymmetric transformations systematically allows us to clarify the relations between existing methods, and gives rise to new ones. In particular, the number of different potentials (subsection 7.5) and of different isospectral potentials (subsection 7.6) obtained by supersymmetry is larger than any previous result. This could be particularly interesting in several physical problems linked with the inverse problem [15].

Acknowledgments

This text presents research results of the Belgian program on interuniversity attraction poles initiated by the Belgian State-Prime Minister's Office Science Policy Programming.

Appendix A

In this appendix, we establish in detail equations (27)–(32) for $(T_l^{(\mathcal{E})}, T_n^{(\mathcal{E})})$, with $\mathcal{E} < 0$. Solutions of H_1 can be expressed in terms of those of H_0 in integral form. For $E \neq \mathcal{E}$, equations (17)–(19), (3) and (26) lead to

$$l_1^{(E)} = \Phi(l_0^{(E)}, l_0^{(\mathcal{E})})/l_0^{(\mathcal{E})} \tag{A1}$$

$$r_1^{(E)} = \Phi(r_0^{(E)}, l_0^{(\mathcal{E})})/l_0^{(\mathcal{E})} \tag{A2}$$

$$\psi_1^{(j)} = |E_j - \mathcal{E}|^{1/2}\Phi(\psi_0^{(j)}, l_0^{(\mathcal{E})})/l_0^{(\mathcal{E})} \tag{A3}$$

$$\psi_1^{(k)} = \mathrm{i}(k - \mathrm{i}\epsilon)\Phi(\psi_0^{(k)}, l_0^{(\mathcal{E})})/l_0^{(\mathcal{E})}. \tag{A4}$$

For $E = \mathcal{E}$, equations (17) and (4), with $x_0 = -\infty$ and $W_0 = 1$, give

$$r_1^{(\mathcal{E})} = (l_0^{(\mathcal{E})})^{-1} \tag{A5}$$

and

$$l_1^{(\mathcal{E})} = \Phi(l_0^{(\mathcal{E})}, l_0^{(\mathcal{E})})/l_0^{(\mathcal{E})}. \tag{A6}$$

Multiplicative constants are introduced in order to simplify equations (A1)–(A3) and (A5). Integrals appearing in these expressions are well defined, even when $l_0^{(\mathcal{E})}$ has nodes.

Between $l_0^{(\mathcal{E})}$ nodes, these functions are solutions of the Schrödinger equation, with potential

$$V_1 = V_0 - 2\frac{d^2}{dx^2}\ln|l_0^{(\mathcal{E})}|. \tag{A7}$$

In the case of (A6), this can be shown by a direct calculation. At $l_0^{(\mathcal{E})}$ nodes, both the potential and its solutions are singular.

The second transformation requires choosing $n_1^{(\mathcal{E})}$. With (A5), equation (23) provides

$$n_1^{(\mathcal{E})} = (l_0^{(\mathcal{E})})^{-1}\left[\alpha + \Phi(l_0^{(\mathcal{E})}, l_0^{(\mathcal{E})})\right] \tag{A8}$$

where $\alpha > 0$. Equation (25) then leads to

$$V_2 = V_0 - 2\frac{d^2}{dx^2}\ln|\alpha + \Phi(l_0^{(\mathcal{E})}, l_0^{(\mathcal{E})})| \tag{A9}$$

which is identical to (27) for (T_l, T_n), and has no singularity. On the other hand, equations (17), (18), and (20) kept in differential form for the second transformation, combined with (A1)–(A4), lead to (28)–(32) for $E \neq \mathcal{E}$. Finally, equations (24) and (A8) give

$$\psi_2^{(\mathcal{E})} = \frac{1}{\alpha^{1/2}}\left[l_0^{(\mathcal{E})} - \frac{l_0^{(\mathcal{E})}\Phi(l_0^{(\mathcal{E})}, l_0^{(\mathcal{E})})}{\alpha + \Phi(l_0^{(\mathcal{E})}, l_0^{(\mathcal{E})})}\right] \tag{A10}$$

which is the extended form of (28)–(32) for $E = \mathcal{E}$. The solutions of V_2 are correct between $l_0^{(\mathcal{E})}$ nodes, but they are also continuous, so that they are correct everywhere. In particular, the singularities of the intermediate potential V_1 and of its solutions are removed by the second transformation.

Appendix B

Here we deduce equations (35) and (36) from equations (33) and (34). Equations (33) and (34) for $m = 0$ are respectively identical to (35) and (36) for $M = 1$. We now prove that if (35) and (36) are valid for some value $M - 1$, they are still valid for the next value M.

Let us apply to potential V_1 a set of modifications at the $M - 1$ energies $\mathcal{E}_2, \ldots, \mathcal{E}_M$. Then, equation (35) implies that V_M reads

$$V_M = V_1 - 2\frac{d^2}{dx^2}\ln|W(\sigma_1^{(\mathcal{E}_2)}, \ldots, \sigma_1^{(\mathcal{E}_M)})| \tag{B1}$$

or

$$V_M = V_0 - 2\frac{d^2}{dx^2}\ln|\sigma_0 W(\sigma_1^{(\mathcal{E}_2)}, \ldots, \sigma_1^{(\mathcal{E}_M)})| \tag{B2}$$

with $\sigma_0 \equiv \sigma_0^{(\mathcal{E}_1)}$. For $m = 0$, equation (34) can be written as

$$\varphi_1^{(E)} = \mathcal{N}_1^{(E)}\left[-\frac{d\varphi_0^{(E)}}{dx} + \frac{\varphi_0^{(E)}}{\sigma_0}\frac{d\sigma_0}{dx}\right] \tag{B3}$$

so that

$$\frac{d^p\varphi_1^{(E)}}{dx^p} = \mathcal{N}_1^{(E)}\left[-\frac{d^{p+1}\varphi_0^{(E)}}{dx^{p+1}} + \frac{\varphi_0^{(E)}}{\sigma_0}\frac{d^{p+1}\sigma_0}{dx^{p+1}}\right] + \sum_{s=0}^{p-1}\left[A_s^p(\sigma_0)\frac{d^s\varphi_1^{(E)}}{dx^s}\right]. \tag{B4}$$

The expressions $A_s^p(\sigma_0)$ are complicated; they contain σ_0 and its p first derivatives only, with $A_0^0 = A_{-1}^0 = 0$ by convention.

The Wronskian in (B2) can be transformed by multiplications and subtractions of its rows into

$$W = \det\left\{\frac{\mathrm{d}^p \sigma_1^{(\mathcal{E}_q)}}{\mathrm{d}x^p} - \sum_{s=0}^{p-1}\left[A_s^p(\sigma_0)\frac{\mathrm{d}^s \sigma_1^{(\mathcal{E}_q)}}{\mathrm{d}x^s}\right]\right\} \tag{B5}$$

with $p = 0, \ldots, M-2$ and $q = 2, \ldots, M$. The normalization of factorization functions has no importance, since a multiplicative constant does not modify the effect of the transformation on the potential. Consequently, $\mathcal{N}_1^{(\mathcal{E}_q)}$ can be taken to be equal to -1. Hence, equations (B4) and (B5) lead to

$$W = \det\left[\frac{\mathrm{d}^{p+1}\sigma_0^{(\mathcal{E}_q)}}{\mathrm{d}x^{p+1}} - \frac{\sigma_0^{(\mathcal{E}_q)}}{\sigma_0}\frac{\mathrm{d}^{p+1}\sigma_0}{\mathrm{d}x^{p+1}}\right] \tag{B6}$$

which, using a determinantal property derived in the appendix of [19], implies

$$W(\sigma_1^{(\mathcal{E}_2)}, \ldots, \sigma_1^{(\mathcal{E}_M)}) = W(\sigma_0^{(\mathcal{E}_1)}, \ldots, \sigma_0^{(\mathcal{E}_M)})/\sigma_0. \tag{B7}$$

Combining equations (B7) and (B2) proves the validity of (35).

The proof of (36) follows the same pattern: the validity of (36) for $M-1$ implies

$$\varphi_M^{(E)} = \left(\prod_{m=1}^{M-1}\mathcal{N}_{m+1}^{(E)}\right)\frac{W(\varphi_1^{(E)}, \sigma_1^{(\mathcal{E}_2)}, \ldots, \sigma_1^{(\mathcal{E}_M)})}{W(\sigma_1^{(\mathcal{E}_2)}, \ldots, \sigma_1^{(\mathcal{E}_M)})}. \tag{B8}$$

The Wronskian elements can be transformed as in (B5) and (B6) into

$$W(\varphi_1^{(E)}, \sigma_1^{(\mathcal{E}_2)}, \ldots, \sigma_1^{(\mathcal{E}_M)}) = \mathcal{N}_1^{(E)} W(\varphi_0^{(E)}, \sigma_0^{(\mathcal{E}_1)}, \ldots, \sigma_0^{(\mathcal{E}_M)})/\sigma_0. \tag{B9}$$

Combining equations (B7), (B8) and (B9) proves equation (36).

Appendix C

Here we deduce equations (39)–(42) from (37) and (38). Equations (37) and (38) for $m = 0$ are identical to (39) and (40) for $M = 1$. We now prove that if (39)–(42) are valid for some value $M-1$, they are still valid for the next value M.

Let us apply to potential V_2 a set of modifications at the $M-1$ energies $\mathcal{E}_2, \ldots, \mathcal{E}_M$. Then, equations (39) and (41) imply

$$V_{2M} = V_2 - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln|\det \mathbf{X}_2| \tag{C1}$$

where $\mathbf{X}_2$ is a matrix of order $M-1$, whose elements for $p, q = 2, \ldots, M$ read

$$X_2^{p,q} = \beta_p \delta_{pq} + \Phi(\sigma_2^{(\mathcal{E}_p)}, \sigma_2^{(\mathcal{E}_q)}). \tag{C2}$$

Equation (38), written for $m = 0$, allows us to calculate, using (26)

$$\Phi(\varphi_2^{(E)}, \chi_2^{(E')}) = \Phi(\varphi_0^{(E)}, \chi_0^{(E')}) - \frac{\Phi(\varphi_0^{(E)}, \sigma_0)\Phi(\chi_0^{(E')}, \sigma_0)}{\beta_1 + \Phi(\sigma_0, \sigma_0)} \tag{C3}$$

where $\varphi_2^{(E)}$ and $\chi_2^{(E')}$ are factorization functions. The determinant property derived in the appendix of [19] implies

$$\det \mathbf{X}_2 = [\beta_1 + \Phi(\sigma_0, \sigma_0)]^{-1} \det \mathbf{X}_0 \tag{C4}$$

where $\mathbf{X}_0$ is defined in (41). Equation (C4), combined with (C1) and (37) for $m = 0$, proves equation (39).

The proof of (40) and (42) follows the same pattern: their validity for $M - 1$ implies that

$$\varphi_{2M}^{(E)} = \left(\prod_{m=1}^{M-1} N_{m+1}^{(E)}\right) \det \mathbf{Y}_2(\varphi_2^{(E)}) / \det \mathbf{X}_2 \tag{C5}$$

where $\mathbf{Y}_2(\varphi_2^{(E)})$ is a matrix of order M, whose elements are defined as in (42), with subscript 0 replaced by 2, and for $p, q = 0, 2, \ldots, M$. Equations (C2), (C3) and (38) imply that

$$Y_2^{p,q}(\varphi_2^{(E)}) = Y_0^{p,q}(\varphi_0^{(E)}) - Y_0^{p,1}(\varphi_0^{(E)}) Y_0^{1,q}(\varphi_0^{(E)}) / Y_0^{1,1}(\varphi_0^{(E)}) \tag{C6}$$

for $p = 2, \ldots, M$ and $q = 0, 2, \ldots, M$. An equation analogous to (C3) with a suitable normalization, equations (C2) and (38) imply that

$$Y_2^{0,q}(\varphi_2^{(E)}) = N_1^{(E)} \left[Y_0^{0,q}(\varphi_0^{(E)}) - Y_0^{0,1}(\varphi_0^{(E)}) Y_0^{1,q}(\varphi_0^{(E)}) / Y_0^{1,1}(\varphi_0^{(E)}) \right] \tag{C7}$$

for $q = 0, 2, \ldots, M$. Using the same determinant property as above, equations (C6) and (C7) lead to

$$\det \mathbf{Y}_2(\varphi_2^{(E)}) = N_1^{(E)} \frac{\det \mathbf{Y}_0(\varphi_0^{(E)})}{\beta_1 + \Phi(\sigma_0, \sigma_0)}. \tag{C8}$$

Combining equations (C4), (C5) and (C8) proves equations (40) and (42).

References

[1] Witten E 1981 *Nucl. Phys.* B **188** 513
[2] Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2917
[3] Sukumar C V 1986 *J. Phys. A: Math. Gen.* **19** 2297
[4] Sukumar C V 1987 *J. Phys. A: Math. Gen.* **20** 2461
[5] Keung W-Y, Sukhatme U P, Wang Q and Imbo T D 1989 *J. Phys. A: Math. Gen.* **22** L987
[6] Grant A K and Rosner J L 1994 *J. Math. Phys.* **35** 2142
[7] Deift P and Trubowitz E 1979 *Commun. Pure Appl. Math.* **32** 121
[8] Luban M and Pursey D L 1986 *Phys. Rev.* D **33** 431
[9] Khare A and Sukhatme U P 1989 *Phys. Rev.* A **40** 6185
[10] Pursey D L 1986 *Phys. Rev.* D **33** 1048
[11] Abraham P B and Moses H E 1980 *Phys. Rev.* A **22** 1333
[12] Thacker H B, Quigg C and Rosner J L 1978 *Phys. Rev.* D **18** 274
[13] Schonfeld J F, Kwong W, Rosner J L, Quigg C and Thacker H B 1980 *Ann. Phys.* **128** 1
[14] Kwong W, Riggs H, Rosner J L and Thacker H B 1989 *Phys. Rev.* D **39** 1242
[15] von Geramb H V (ed) 1993 *Quantum Inversion Theory and Applications (Lecture Notes in Physics 427)* (Berlin: Springer)
[16] Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2937
[17] Baye D 1987 *Phys. Rev. Lett.* **58** 2738
[18] Baye D 1987 *J. Phys. A: Math. Gen.* **20** 5529
[19] Ancarani L U and Baye D 1992 *Phys. Rev.* A **46** 206
[20] Baye D 1993 *Phys. Rev.* A **48** 2040
[21] Baye D 1993 *Quantum Inversion Theory and Applications (Lecture Notes in Physics 427)* ed H V von Geramb (Berlin: Springer) p 127
[22] Pappademos J, Sukhatme U and Pagnamenta A 1993 *Phys. Rev.* A **48** 3525
[23] Baye D and Sparenberg J-M 1994 *Phys. Rev. Lett.* **73** 2789
[24] Hartman P 1982 *Ordinary Differential Equations* (Boston, MA: Birkhäuser) 2nd edn
[25] Chaturvedi S and Raghunathan K 1986 *J. Phys. A: Math. Gen.* **19** L775
[26] Khare A and Sukhatme U 1989 *J. Phys. A: Math. Gen.* **22** 2847
[27] Pursey D L and Weber T A 1994 *Phys. Rev.* A **50** 4472